Exam answers Asset Pricing Theory June 2011

Academic Aims Asset Pricing Theory

Modern finance theory analyses investment decisions under uncertainty with regard to the pricing of traded assets including derivatives. Throughout the course students will acquire an introductory understanding of the main elements in modern financial theory and the ability to apply the different models and techniques to problems of a limited complexity. Students are also expected to obtain an understanding of the mathematical methods related to the theories of finance including selected proofs and the associated practical methods.

The excellent performance is characterized by a good knowledge of the theories, methods, models and proofs covered in the course together with the ability to apply these competencies to explicit problems, in particular within:

Discrete time models

The one-period-model under uncertainty, equilibrium prices, the binomial model, arbitrage-pricing theory (ATP) and the multi-period-model.

Continuous times models

Wiener processes and Ito's lemma; Black-Scholes-Mertons model; martingales and measures; term structure models, models of the short rate, the Heath, Jarrow and Morton model (HJM); credit risk.

Problem 1 Pricing of derivative security

(a) We are given the function $G(S_t, t) = (S_t)^3$, finding the derivatives with respect to t, S and the double derivative with respect to S gives us:

$$\frac{\partial G(S_t,t)}{\partial t} = 0, \frac{\partial G(S_t,t)}{\partial S_t} = 3(S_t)^2 \text{ and } \frac{\partial^2 G(S_t,t)}{\partial S_t^2} = 6S_t$$

Inserting in Itô's lemma it gives us

$$dG(S_t, t) = 3(\mu + \sigma^2)S_t^3 dt + 3\sigma S_t^3 dz_t = 3G(S_t, t)(\mu + \sigma^2)dt + 3G(S_t, t)\sigma dz_t$$

i.e. a geometric Brownian motion

(b) Risk neutral valuation means that the price of the derivative is equal to the discounted value of the expected pay off $H(S_t, t) = e^{-r(T-t)}E_t^Q[S_T^3]$ where Q is the risk free probability measure.

We recall that under the probability measure Q, the drift of the stock will be r (i.e. μ=r) and that if Y follows a geometrical Brownian motion dY = aYdt + sYdz then $E_t[Y_T] = Y_t e^{a(T-t)}$ then

$$H(S_t,t) = e^{-r(T-t)} S_t^3 e^{(3r+3\sigma^2)(T-t)} = S_t^3 e^{(2r+3\sigma^2)(T-t)}$$

(c) First we differentiate H, and then insert into Black-Scholes-Merton differential equation

$$\frac{\partial H(S_t,t)}{\partial t} = -(2r+3\sigma^2)H(S_t,t), \frac{\partial H(S_t,t)}{\partial S_t} = \frac{3H(S_t,t)}{S_t} \; and \; \frac{\partial^2 H(S_t,t)}{\partial S_t^2} = \frac{6H(S_t,t)}{S_t^2}$$

$$-(2r+3\sigma^2)H(S_t,t) + 3rH(S_t,t) + 3\sigma^2 H(S_t,t) - rH(S_t,t) =$$

$$((-2+3+0-1)r + (-3+0+3-0)\sigma)H(S_t,t) = 0$$

Problem 2 — Short term interest rate

(a) Since r_t is compounded continuously, to the amount on our bank account β_t then the change in the bank account must be $d\beta_t = r_t \beta_t dt$ on integral form $\beta_t = \beta_0 e^{\int_0^t r_s ds}$

(b) Given risk neutral valuation the value of the zero coupon bond will be

$$P(t,T) = E_t^Q\left[e^{-\int_t^T r_s ds}\right] = E_t^Q\left[\frac{\beta_t}{\beta_T}\right] = \beta_t E_t^Q\left[\frac{1}{\beta_T}\right]$$

If one rather wants to use the bank account for the expression. Now it is introduced.

(c) In a general one factor model r_t would follow a Itô process

$$dr_t = \mu(r_t,t)dt + \sigma(r_t,t)dz_t$$

where z_t is a generalized Wiener process.

The two most obvious examples given the text syllabus would be

Vasiček: where $\mu(r_t,t) = b - ar_t = \kappa(\theta - r_t)$ and $\sigma(r_t,t) = \sigma$

Or CIR: where $\mu(r_t,t) = b - ar_t = \kappa(\theta - r_t)$ and $\sigma(r_t,t) = \sigma\sqrt{r_t}$

(d) If the term structure of interest is affine, then the zero coupon prices can be written on the form $P(t,T) = A(T-t)e^{-B(T-t)r_t}$

If we apply Itô's lemma to P, we get that

$$dP(t,T) = (\cdots)dt + \frac{\partial P(t,T)}{\partial r_t}\sigma(r_t,t)dz_t = (\cdots)dt - B(T-t)\,\sigma(r_t,t)P(t,T)dz_t$$

Since we are under the risk neutral probability measure, the drift of the bond has to be r_t.

And the volatility term equals the product of the two elements as described in the problem.

That P(T,T) = 1 is obvious and can easily be seen from (b)

Problem 3 — Credit derivatives

(a) The hazard rate λ is default intensity of a given corporate issuing debt.

The probability of default from t to T is then $Q(\lambda, T-t) = (1 - e^{-\lambda(T-t)})$

The recovery rate R is the percentage of the notional amount an investor in a corporate bond receives at maturity if the issuer defaults before maturity.

The risk free is used to discount cash flows, can e.g. be expressed as zero coupon prices P(t,T)

The price of the corporate bond is then

$$Corp(\lambda, t, T) = P(t,T)(Q(\lambda, T-t) * R + (1 - Q(\lambda, T-t)))$$

(b) The prices are under the risk free measure Q means it shows the probabilities of events and prices of assets as if investors where risk neutral. The difference to the actual risk measure P ensures that investors get a risk premium.

(c) CB^{A}=0.75 * (20% * 40% + (1-20%)) = 0.75 * (88%) = 0.66

CB^{B}=0.75 * (40% * 40% + (1-40%)) = 0.75 * (76%) = 0.57

(d) A CDS is a contract where the buyer of protection is insured against default of the underlying entity. In case of default will the CDS buyer, have the right to deliver the underlying corporate bond to the seller of the CDS to the price of the notional.

If we buy a CDS on company A, and add it to the corporate bond, we are certain to get 1 at time T, this portfolio has to have the same value as the zero coupon bond, i.e. 0.75.

The price of the CDS^{A} = 0.75 -0.66 = 0.09 and CDS^{B} = 0.75 -0.66 = 0.18

(e) If we construct a portfolio of bought CDS's on company A and B (CDS^{A} + CDS^{B}) and sold the 1st and 2nd to default swap on the basket of A and B (CDS^{1st} + CDS^{2nd}).

Then if none of the companies defaults, no payout has come from the CDS's

If one of the companies defaults there will be a payment in CDS^{A} or CDS^{B} depending on which one defaults and a opposite payment from the CDS^{1st}.

If both companies default then there will be payments from CDS^{A} and CDS^{B} the payments will be offset by CDS^{1st} and CDS^{2nd} since we now have pay off from both of these contracts.

Since this portfolio is risk free, the value of (CDS^{1st} + CDS^{2nd}) = (CDS^{A} + CDS^{B}) to avoid arbitrage.

The price of CDS^{nth} will be probability of pay out* (1 – recovery rate) * discount factor

(f) If we look at sets, then we have Def^A is the set where company A defaults, and Def^B is the set where company B defaults, and $Pr(Def^A) = Q^A = 20\%$, $Pr(Def^B) = Q^B = 40\%$. The set where the second to default swap pays out is then the $Def^A \cap Def^B$ and the set where the first to default swap pays out is then the $Def^A \cup Def^B$. To find the maximum, minimum prices we have to maximize and minimize the probability of ending in this sets.

We start by minimising the price of CDS^{2nd} the same as minimising $Pr(Def^A \cap Def^B)$. We get that it is possible to have a situation where maximum one of the companies defaults. This gives $CDS^{2nd} = 0$, the CDS^{1nd}=0.27 = 60% * (1 – 40%) * 0.75. According to the no arbitrage condition above and to the pricing formula in the problem text. The probabilities of default is given in the table below, where we have to secure that the probability of default for either company A or B equals the given probabilities above.

Probabilities to minimum price CDS(2^{nd})		
Comp. B ↓/ A→	Survives	Defaults
Survives	40%	20%
Defaults	40%	0%

Probabilities to maximum price CDS(2^{nd})		
Comp B ↓/ A→	Survives	Defaults
Survives	60%	0%
Defaults	20%	20%

Similiar the maximum price of CDS^{2nd} can be found, by getting $Pr(Def^A \cap Def^B)$ be as high as possible. This happens if Company A only defaults if Company B also defaults. See table above to the right. CDS^{2nd} gets same price as CDS^A and CDS^{1st} gets same price as CDS^B

CDS^{2nd} = 0.09 = 20% * (1 – 40%) * 0.75 and the CDS^{1nd}=0.18 = 40% * (1 – 40%) * 0.75.

(g) with the new probabilities we can update the table

Probabilities to minimum price CDS(2^{nd})		
Comp. B ↓/ A→	Survives	Defaults
Survives	0%	40%
Defaults	60%	0%

Probabilities to maximum price CDS(2^{nd})		
Comp B ↓/ A→	Survives	Defaults
Survives	40%	0%
Defaults	20%	40%

CDS^{2nd} = 0.00 = 0% * (1 – 40%) * 0.75 and the CDS^{1nd}=0.45 = 100% * (1 – 40%) * 0.75.

CDS^{2nd} = 0.18 = 40% * (1 – 40%) * 0.75 and the CDS^{1nd}=0.27 = 60% * (1 – 40%) * 0.75.

(h) What happened in the recent financial crisis was that a lot of structured credit was issued, with arguments of low default correlation. At the same time the credit spread was relatively low. We can compare it to the minimum price of the 2^{nd} to default swap in (f) It would be relatively cheap to buy protection against a lot of companies would default at the same time.

When the crisis started default probabilities rose, and it became obvious that the default probability would be higher for some type of credits, especially subprime mortgages. This leads us to maximum price of the second to default swap in (g), where the price on protecting this scenario suddenly has increased 18% of the notional, a huge loss on something that seemed to be risk free.